MATH2020A Solutions of Midterm Exam

1. (a) (i) $-|f| \le f \le |f| \Rightarrow |f| \pm f \geq 0$,

by the positivity of integration and linearity, we have

$0 \le \iint_R (|f| \pm f)\, dA = \iint_R |f|\, dA \pm \iint_R f\, dA$

$\Rightarrow \pm \iint_R f\, dA \leq \iint_R |f|\, dA \Rightarrow \left| \iint_R f\, dA \right| \leq \iint_R |f|\, dA.$

(ii) Since $M - f \ge 0$ and $f - m \ge 0$,

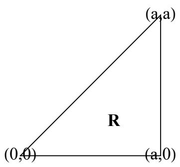
by the positivity of integration and linearity, we have

$\iint_R (M - f)\, dA \ge 0$ and $\iint_R (f - m)\, dA \ge 0$

$\Rightarrow m \text{ Area } (R) = \iint_R m\, dA \le \iint_R f\, dA \le \iint_R M\, dA = M \text{ Area } (R)$

$\Rightarrow m \le \dfrac{1}{\text{Area } (R)} \iint_R f\, dA \le M.$

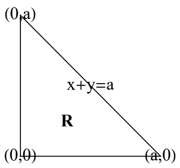
(b)



$\int_0^a \int_0^x f(x, y)\, dy\, dx = \iint_R f(x, y)\, dA = \int_0^a \int_y^a f(x, y)\, dy\, dx$

by Fubini ' s theorem.

(c)



Let $u = x + y$, $v = y$, then $0 \le u \le a$, $0 \le v = u - x \le u$ and $\dfrac{\partial(x, y)}{\partial(u, v)} = \left| \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right| = 1$. Thus,

$\iint_R f(x + y)\, dA = \int_0^a \int_0^u f(u)\, dv\, du = \int_0^a u f(u)\, du$

2. $(x^2 + y^2)^3 = 4(x^4 + y^4) \Rightarrow$ use polor coordinates, $r^6 = 4 r^4 (\cos^4 \theta + \sin^4 \theta)$,

$\Rightarrow r^2 = 4(\cos^4 \theta + \sin^4 \theta)$ since $r \neq 0$. Thus,

$\text{Area} = \int_0^{2\pi} \int_0^{2\sqrt{\cos^4 \theta + \sin^4 \theta}} r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{r^2}{2} \right]_0^{2\sqrt{\cos^4 \theta + \sin^4 \theta}} d\theta = \int_0^{2\pi} 2(\cos^4 \theta + \sin^4 \theta)\, d\theta$

$= 2 \int_0^{2\pi} (\cos^4 \theta + \sin^4 \theta + 2\cos^2 \theta \sin^2 \theta - 2\cos^2 \theta \sin^2 \theta)\, d\theta$

$= 2 \int_0^{2\pi} \left(1 - \frac{1}{2} \sin^2 (2\theta)\right) d\theta = 4\pi - \int_0^{2\pi} \frac{1 - \cos(4\theta)}{2}\, d\theta = 3\pi.$

3. The solid is bounded above by $z = 1 - \left(\frac{x^2}{4} + y^2\right)^2$ and below by $z = 0$. Thus, the volume $V = \iint_{\left(\frac{x^2}{4}+y^2\right)^2 \le 1} 1 - \left(\frac{x^2}{4} + y^2\right)^2 \, dx\, dy$. Let $u = \frac{x}{2}$, $v = y$, then $\left(\frac{x^2}{4} + y^2\right)^2 \le 1$ is equivalent to $u^2 + v^2 \le 1$ and $\frac{\partial(x, y)}{\partial(u, v)} = \left| \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \right| = 2$. Thus,

$$V = \iint_{(u^2+v^2)^2 \le 1} \left(1 - (u^2 + v^2)^2\right) \times 2 \, du\, dv$$
$$= (\text{use polor coordinates}) \; 2 \int_0^{2\pi} \int_0^1 \left(1 - r^4\right) r \, dr\, d\theta$$
$$= 2\pi \int_0^1 \left(1 - r^4\right) dr^2 = 2\pi \left[r^2 - \frac{1}{3} r^6\right]_0^1 = \frac{4\pi}{3}.$$

4. By the spherical coordinates $\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$, $D = \begin{cases} 0 \le \rho \le a \\ 0 \le \phi \le \frac{\pi}{2} \\ 0 \le \theta \le \frac{\pi}{2} \end{cases}$. Thus,

$$\iiint_D xyz \, dV = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a (\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)(\rho \cos\phi)(\rho^2 \sin\phi) \, d\rho\, d\phi\, d\theta$$
$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a \rho^5 \sin^3\phi \cos\theta \, \sin\theta \cos\phi \, d\rho\, d\phi\, d\theta$$
$$= \left(\int_0^a \rho^5 \, d\rho\right) \left(\int_0^{\frac{\pi}{2}} \sin^3\phi \cos\phi \, d\phi\right) \left(\int_0^{\frac{\pi}{2}} \sin\theta \cos\theta \, d\theta\right)$$
$$= \left[\frac{\rho^6}{6}\right]_0^a \left[\frac{\sin^4\phi}{4}\right]_0^{\frac{\pi}{2}} \left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{a^6}{48}.$$

5. $Q(x, y) =$

$$a x^2 + 2 b x y + c y^2 + 2 d x + 2 e y + f = a x^2 + \frac{2 b}{\sqrt{a}} y \left(\sqrt{a}\, x\right) + c y^2 + 2 d x + 2 e y + f \quad (\text{since } a > 0)$$

$$= \left(\sqrt{a}\, x + \frac{b}{\sqrt{a}} y\right)^2 + \frac{a c - b^2}{a} y^2 + 2 d x + 2 e y + f.$$

Let $w = \sqrt{a}\, x + \frac{b}{\sqrt{a}} y$, then $x = \frac{w}{\sqrt{a}} - \frac{b}{a} y$ and

$$Q = w^2 + \frac{a c - b^2}{a} y^2 + 2 d \left(\frac{w}{\sqrt{a}} - \frac{b}{a} y\right) + 2 e y + f \quad (\text{since } ac - b^2 > 0)$$

$$= \left(w + \frac{d}{\sqrt{a}}\right)^2 - \frac{d^2}{a} + \left(\sqrt{\frac{a c - b^2}{a}}\, y\right)^2 +$$

$$\frac{2 (a e - b d)}{a} \frac{\sqrt{a}}{\sqrt{a c - b^2}} \sqrt{\frac{a c - b^2}{a}}\, y + \frac{(a e - b d)^2}{a \left(a c - b^2\right)} - \frac{(a e - b d)^2}{a \left(a c - b^2\right)} + f$$

$$= \left(w + \frac{d}{\sqrt{a}}\right)^2 - \frac{d^2}{a} + \left(\sqrt{\frac{a c - b^2}{a}}\, y + \frac{(a e - b d)}{\sqrt{a}\sqrt{a c - b^2}}\right)^2 - \frac{(a e - b d)^2}{a \left(a c - b^2\right)} + f.$$

Let $u = w + \frac{d}{\sqrt{a}} = \sqrt{a}\, x + \frac{b}{\sqrt{a}} y + \frac{d}{\sqrt{a}}$, $v = \sqrt{\frac{a c - b^2}{a}}\, y + \frac{(a e - b d)}{\sqrt{a}\sqrt{a c - b^2}}$, then

$$Q = u^2 + v^2 - \frac{d^2}{a} - \frac{(a e - b d)^2}{a \left(a c - b^2\right)} + f \text{ and } \frac{\partial (u, v)}{\partial (x, y)} = \left| \begin{pmatrix} \sqrt{a} & \frac{b}{\sqrt{a}} \\ 0 & \sqrt{\frac{a c - b^2}{a}} \end{pmatrix} \right| = \sqrt{a c - b^2} \neq 0. \text{ Therefore,}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-Q(x,y)}\, dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2 - v^2 + \frac{d^2}{a} + \frac{(a e - b d)^2}{a \left(a c - b^2\right)} - f} \frac{1}{\sqrt{a c - b^2}}\, du\, dv$$

$$= e^{\frac{d^2}{a} + \frac{(a e - b d)^2}{a (a c - b^2)} - f} \frac{1}{\sqrt{a c - b^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2 - v^2}\, du\, dv = e^{\frac{d^2}{a} + \frac{(a e - b d)^2}{a (a c - b^2)} - f} \frac{1}{\sqrt{a c - b^2}} \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta$$

$$= e^{\frac{d^2}{a} + \frac{(a e - b d)^2}{a (a c - b^2)} - f} \frac{1}{\sqrt{a c - b^2}} \pi \int_0^{\infty} e^{-r^2}\, dr^2 = e^{\frac{d^2}{a} + \frac{(a e - b d)^2}{a (a c - b^2)} - f} \frac{\pi}{\sqrt{a c - b^2}}.$$